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ASYMPTOTIC SOLUTION TO THE PROBLEM OF AN ELASTIC BODY LYING ON SEVERAL SMALL SUPPORTS†

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The equilibrium of an elastic body with a plane base placed on several small smooth rigid supports $\Gamma(\varepsilon)$ is studied. One-sided contact boundary conditions are imposed on $\Gamma(\varepsilon)$ The leading terms of the asymptotic solution of the problem as $\varepsilon \to 0$ are constructed and justified; the problem becomes statically indeterminate when the number of supports exceeds three. The problem of finding the contact zone reduces to solving a non-linear algebraic problem. Besides the three equilibrium equations which connect the unknown support reactions, this problem includes compatibility of the deformation conditions which contain, in particular, three unknown parameters describing the settlement of the body. Necessary and sufficient conditions for the existence and uniqueness of the solution of the limiting algebraic problem are proved.

1. STATEMENT OF THE PROBLEM

Suppose that in its undeformed state the body occupies a domain Ω in the space $\mathbf{R}^3$ with boundary $\partial\Omega$, a part Σ of which coincides with part of the $x_3=0$ plane (see Fig. 1). On $\Sigma\backslash\partial\Sigma$ we select points $P^1, \ldots, P^J$ with coordinates $(x_1^j, x_2^j, 0)$, $j=1, \ldots, J$. Here and below our notation does not distinguish between points in $\mathbf{R}^2$ and their images in the $\{\mathbf{x}: x_3=0\}$ plane in $\mathbf{R}^3$. Suppose also that ω_j is a domain in $\mathbf{R}^2$ bounded by a simple smooth closed contour $\partial\omega_j$, ε is a small positive parameter, and

$$\omega_j(\varepsilon)=\{(x_1,x_2,0)\in\Sigma\setminus\partial\Sigma:\varepsilon^{-1}(x_1-x_1^j,x_2-x_2^j)\in\omega_j\};\quad j=1,\ldots,J$$

Additional assumptions will be made in Sections 3 and 4 about the relative positions of the points P^j and the sets $\omega_j(\varepsilon)$. The union of all the closures $\overline{\omega_j(\varepsilon)}$ is denoted by $\Gamma(\varepsilon)$. Under the action of volume forces $\mathbf{f}$ the body can rest on plane completely rigid smooth supports $\Gamma(\varepsilon)$. The surface $\partial\Omega\setminus\Gamma(\varepsilon)$ is assumed to be unloaded.

In order to write the elasticity equations in a convenient from we will use the following notation

$$\mathbf{\Phi}^i(\mathbf{x})=\mathbf{e}^i x_i,\ \mathbf{\Phi}^4(\mathbf{x})=2^{-1/2}(x_2\mathbf{e}^3+x_3\mathbf{e}^2) \tag{1.1}$$

$$\mathbf{\Phi}^5(\mathbf{x})=2^{-1/2}(x_1\mathbf{e}^3+x_3\mathbf{e}^1),\ \ \mathbf{\Phi}^6(\mathbf{x})=2^{-1/2}(x_2\mathbf{e}^1+x_1\mathbf{e}^2)$$

$$\boldsymbol{\varphi}^i=\mathbf{e}^i,\quad \boldsymbol{\varphi}^{3+i}(\mathbf{x})=2^{-1/2}\mathbf{x}\times\mathbf{e}^i \tag{1.2}$$

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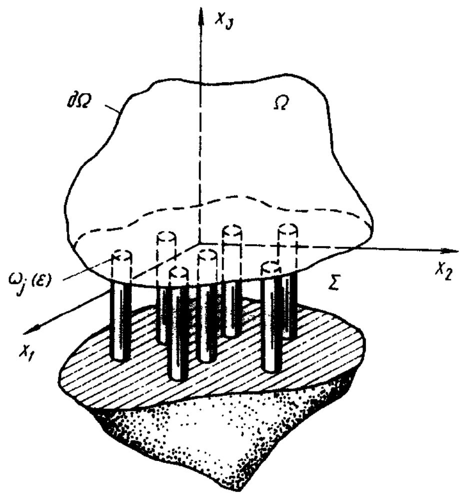

Fig. 1.

Here $i=1,2,3$, $\mathbf{e}^i$ is the unit vector along the x_i axis, the cross denotes the vector product, and all vectors are expressed as columns. Suppose also that $D(\mathbf{x})$ is a (3×6) matrix with columns $\Phi^1(\mathbf{x}), \dots, \Phi^6(\mathbf{x})$ and that $d(\mathbf{x})$ is a (3×6) matrix with columns $\varphi^1, \dots, \varphi^6(\mathbf{x})$, and that A is a (6×6) matrix of the elastic moduli for an isotropic body

$$A_{jk} = \lambda + 2\mu\delta_{jk},\ A_{pq} = \mu\delta_{pq}$$
$$A_{jq} = A_{pk} = 0,\ j,k = 1,2,3;\ p,q = 4,5,6$$

where δ_{jk} is the Kronecker delta and λ and μ are Lamé coefficients.

If $\mathbf{u} = (u_1, u_2, u_3)^t$ is the displacement vector, t denotes transposition, and ∇_x is the gradient, then the six-dimensional columns

$$D(\nabla_x)^t\mathbf{u}(\mathbf{x}),\ \boldsymbol{\sigma}(\mathbf{u}(\mathbf{x})) = AD(\nabla_x)^t\mathbf{u}(\mathbf{x})$$

are the strain and stress vectors. For example, the stress vector is expressed in terms of the Cartesian components σ_{ij} of the stress tensor as follows:

$$\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, 2^{-1/2}\sigma_{23}, 2^{-1/2}\sigma_{31}, 2^{-1/2}\sigma_{12})^t$$

It can be shown that the Lamé system and the homogeneous boundary conditions on $\partial\Omega\setminus\Gamma(\varepsilon)$ can be written as

$$L(\nabla_x)\mathbf{u}(\varepsilon,\mathbf{x}) \equiv -D(\nabla_x)AD(\nabla_x)^t\mathbf{u}(\varepsilon,\mathbf{x}) = \mathbf{f}(\mathbf{x}),\ \mathbf{x}\in\Omega \tag{1.3}$$
$$B(x,\nabla_x)\mathbf{u}(\varepsilon,\mathbf{x}) \equiv D(\boldsymbol{\nu}(\mathbf{x}))AD(\nabla_x)^t\mathbf{u}(\varepsilon,\mathbf{x}) = 0,\ \mathbf{x}\in\partial\Omega\setminus\Gamma(\varepsilon)$$

Here ν is the unit vector of the inner normal to the boundary of the domain Ω. Equation (1.3) are closed by one-sided contact conditions at the sets $\omega_j(\varepsilon)$

$$\sigma_{31}(\mathbf{u};\varepsilon,\mathbf{x}) = \sigma_{32}(\mathbf{u};\varepsilon,\mathbf{x}) = 0 \tag{1.4}$$
$$u_3(\varepsilon,\mathbf{x}) \geqslant 0,\ \sigma_{33}(\mathbf{u};\varepsilon,\mathbf{x}) \leqslant 0$$
$$u_3(\varepsilon,\mathbf{x})\sigma_{33}(\mathbf{u};\varepsilon,\mathbf{x}) = 0,\quad \mathbf{x}\in\omega_j(\varepsilon);\ j = 1,\dots,J \tag{1.5}$$

We know (see [1, 2], etc.) that the Signorini problem (1.3)–(1.5) is equivalent to finding the minimum of the potential energy functional $\mathbf{J}(\mathbf{v}) = a(\mathbf{v}, \mathbf{v}) - F(\mathbf{v})$ on the convex cone

$$K = \{\mathbf{v} \in H^1(\Omega): \upsilon_3(\mathbf{x}) \geqslant 0, \mathbf{x} \in \Gamma(\varepsilon)\}$$

$H^1(\Omega)$ denotes the space of vector functions with finite elastic energy

$$a(\mathbf{v},\mathbf{v}) \equiv \tfrac{1}{2}(AD(\nabla_x)^t \mathbf{v}, D(\nabla_x)^t \mathbf{v})_\Omega$$

and $F(\mathbf{v}) \equiv (\mathbf{f}, \mathbf{v})_\Omega$ is the work done by body forces $\mathbf{f} \in L_2(\Omega)$ during admissible displacements $\mathbf{v} \in K$; here $(\,,)_\Omega$ is the scalar product in $L_2(\Omega)$.

The case of balanced loads is not considered. The equilibrium equations for a body on smooth supports require that

$$F_1(1) = F_2(1) = 0,\ F_2(x_1) - F_1(x_2) = 0,\ F_3(1) < 0 \tag{1.6}$$

Here $F_i(\gamma) \equiv (\mathbf{f}, \gamma\mathbf{e}^i)_\Omega$ for $\gamma \in L_2(\Omega)$.

In this case the system of forces is statically equivalent to a single resultant force $F_3(1)\mathbf{e}^3$ applied to an arbitrary point on the central axis of the system

$$x_1 = x_1^0 \equiv F_3(1)^{-1}(F_3(x_1) - F_1(x_3))$$
$$x_2 = x_2^0 \equiv F_3(1)^{-1}(F_3(x_2) - F_2(x_3))$$

The intersection of all the closed half-planes containing $\Gamma(\varepsilon)$ is referred to as $\Phi(\varepsilon)$, the convex shell of $\Gamma(\varepsilon)$. The polygon $\Gamma(0)$ is denoted by $\mathbf{P}$. Let $\mathbf{R} = \{r : r = d(\mathbf{x})a,\ a \in \mathbf{R}^6\}$ be the space of solid displacements, and $\mathbf{R}' = \mathbf{R} \cap K$, and $\mathbf{R}'' = \{r \in \mathbf{R}' : -r \in \mathbf{R}'\}$ be the subset of $\mathbf{R}'$ generated by all "two-sided" displacements $r_1 = c_1 - c_0 x_2,\ r_2 = c_2 + c_0 x_1,\ r_3 = 0$.

The following conditions are necessary and sufficient [1, Section 2.10] for the existence of an absolute minimum for the functional $\mathbf{J}$ on K: the central axis of the applied system of forces intersects the set $\Gamma(\varepsilon)$ at an internal point. Under the assumptions made above the vector function from K minimizing $\mathbf{J}$, if it exists, is defined, apart from an arbitrary two-sided displacement.

In this paper the asymptotic behaviour of the solution of the Signorini problem (1.3)–(1.5) as $\varepsilon \to 0$ is constructed by the method of matched asymptotic expansions. We emphasize that a simple passage to the limit is impossible: when $\varepsilon = 0$ relations (1.4) and (1.5) disappear, and Eqs (1.3) are transformed into the second fundamental boundary-value problem of the theory of elasticity

$$L(\nabla_x)\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \mathbf{x} \in \Omega;\ B(\mathbf{x}, \nabla_x)\mathbf{u}(\mathbf{x}) = 0, \mathbf{x} \in \partial\Omega \tag{1.7}$$

In the case of (1.6) this problem has no solutions.

2. CONSTRUCTION OF THE ASYMPTOTIC FORM

We will use the method of matched asymptotic expansions (see [3, 4] etc.), and look for two types of expansion: an outer one, valid far from the set $\Gamma(\varepsilon)$

$$\mathbf{u}(\varepsilon, \mathbf{x}) = \varepsilon^{-1}\mathbf{v}^0(\mathbf{x}) + \varepsilon^0 \mathbf{v}^1(\mathbf{x}) + \ldots \tag{2.1}$$

and an inner one, valid in small neighbourhoods of $\omega_j(\varepsilon)$

$$\mathbf{u}(\varepsilon, \mathbf{x}) = \varepsilon^{-1}\mathbf{w}^{0j}(\xi^j) + \varepsilon^0 \mathbf{w}^{1j}(\xi^j) + \ldots \tag{2.2}$$

In (2.2) we have introduced the "stretched" variables

$$\xi^j = (\xi_1^j, \xi_2^j, \xi_3^j), \quad \xi^j = \varepsilon^{-1}(\mathbf{x} - P^j) \tag{2.3}$$

Substituting (2.1) into (1.3), we find that $\mathbf{v}^0$ is a solid displacement in $\mathbf{R}\setminus\mathbf{R}''$, i.e. $\mathbf{v}^0 = d(\mathbf{x})a^0$, $a^0 \in \mathbf{R}^6$, $a_i^0 = 0$ $(i = 1, 2, 6)$. Furthermore, the function $\mathbf{v}^1$ should satisfy problem (1.7). The method of matched asymptotic expansions assumes that the terms $\varepsilon^{k-1}\mathbf{v}^k$ from the right-hand side of (2.1) can have singularities of orders $O(|x - P^j|^{-k})$ at the points P^j (near the perturbation zone of the boundary conditions). This allows the expression for $\mathbf{v}^1$ to have singularities of order $O(|x - P^j|^{-1})$ generated by the point forces. We denote by $\mathbf{G}^j$ the generalized Green's function which satisfies the relation

$$L(\nabla_x)\mathbf{G}^j(\mathbf{x}) = -d(\mathbf{x})c^j, \mathbf{x} \in \Omega; \mathbf{B}(\mathbf{x}, \nabla_x)\mathbf{G}^j(\mathbf{x}) = 0, \mathbf{x} \in \partial\Omega \setminus P^j \tag{2.4}$$

$$(\mathbf{G}^j, \varphi^i)_\Omega = 0, i = 3,4,5; \mathbf{G}^j(x) = \mathbf{T}(\mathbf{x} - P^j) + O(1), \mathbf{x} \to P^j$$

The notation is as follows: $\mathbf{T}$ is the solution of the Boussinesq problem of the loading of an elastic half-space $x_3 \geq 0$ by a unit point force applied at the origin of coordinates and directed along the x_3 axis

$$4\pi\mu\mathbf{T}_i(\mathbf{x}) \equiv x_i x_3 |\mathbf{x}|^{-3} + (1 - \alpha^{-1})x_i|\mathbf{x}|^{-1}(|\mathbf{x}| + x_3)^{-1}, i = 1,2$$

$$4\pi\mu\mathbf{T}_3(\mathbf{x}) \equiv x_3^2|\mathbf{x}|^{-3} + \alpha^{-1}|\mathbf{x}|^{-1}, \alpha \equiv (\lambda + 2\mu)^{-1}(\lambda + \mu)$$

The vector c^j is determined from the system

$$Sc^j = b^j = (0,0,1,2^{-1/2}x_2^j, -2^{-1/2}x_1^j, 0)^t \tag{2.5}$$

We recall that the Gram matrix $S = \|(\varphi^i, \varphi^k)_\Omega\|_{i,k=1}^6$ is non-singular.

Suppose also that $\mathbf{v}^{10} \in H^2(\Omega)$ is a solution of problem (1.7) for a self-equilibrating load

$$\mathbf{f}^0(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - d(\mathbf{x})c^0 \tag{2.6}$$

with $(\mathbf{v}^{10}, \varphi^i)_\Omega = 0$ $(i = 3, 4, 5)$. The vector c^0 satisfies system (2.5) with right-hand side components $b_i^0 = (\mathbf{f}, \varphi^i)_\Omega$ $(i = 1, \ldots, 6)$. Then we can have the following representation for $\mathbf{v}^1$

$$\mathbf{v}^1(\mathbf{x}) = \mathbf{v}^{10}(\mathbf{x}) + d(\mathbf{x})a^1 + R_1\mathbf{G}^1(\mathbf{x}) + \ldots + R_J\mathbf{G}^J(\mathbf{x}) \tag{2.7}$$

Here $d(\mathbf{x})a^1 \in \mathbf{R}\setminus\mathbf{R}''$, and the R_j are certain constants.

Because the vectors $\mathbf{v}^{10}$ and $\mathbf{G}^j$ leave errors in the system of equilibrium equations (of the form $d(\mathbf{x})c^h$, see (2.6) and (2.4)), the vector (2.7) satisfies (1.7) only when the additional conditions

$$-F_3(1) = R_1 + \ldots + R_J \tag{2.8}$$

$$-x_1^\circ F_3(1) = x_1^1 R_1 + \ldots + x_1^J R_J, \quad -x_2^\circ F_3(1) = x_2^1 R_1 + \ldots + x_2^J R_J$$

are satisfied.

We will now consider the construction of the inner expansion terms, with the help of which condition (1.6) will be satisfied. We change to the fast variables (2.3) and then put $\varepsilon = 0$. As a result we obtain from (1.4)–(1.6) the model Signorini equation for determining $\mathbf{w}^{0j}$

$$L(\nabla_\xi)\mathbf{w}^{0j}(\xi) = 0, \xi \in \mathbf{R}_+^3 = \{\xi \in \mathbf{R}^3 : \xi_3 > 0\} \tag{2.9}$$

$$\sigma_{31}(\mathbf{w}^{0j}(\xi', 0)) = \sigma_{32}(\mathbf{w}^{0j}(\xi', 0)) = 0, \xi' \equiv (\xi_1, \xi_2) \equiv \mathbf{R}^2$$

$$\mathbf{w}_3^{0j}(\xi',0) \geq 0, \sigma_{33}(\mathbf{w}^{0j}(\xi',0)) \leq 0$$
$$\mathbf{w}_3^{0j}(\xi',0)\sigma_{33}(\mathbf{w}^{0j}(\xi',0)) = 0, \xi' \in \omega_j \tag{2.10}$$

We will omit the superscripts on the symbol ξ^j. Relations (2.9) and (2.10) are supplied with additional asymptotic conditions obtained by matching the outer and inner expansions. Because $|\xi^j| = \varepsilon^{-1}|\mathbf{x} - P^j|$, by extracting the leading asymptotic forms from $\mathbf{v}^0$ and $\mathbf{v}^1$ we obtain

$$\varepsilon^{-1}\mathbf{v}^0(\mathbf{x}) + \mathbf{v}^1(\mathbf{x}) \sim \varepsilon^{-1}\{d(P^j)a^0 + R_j\mathbf{T}(\xi^j)\} \tag{2.11}$$

as $\mathbf{x} \to P^j$.

Comparing expansions (2.1) and (2.2) of the same function $\mathbf{u}$ in the $\{\mathbf{x} : c\varepsilon^{1/2} \leq |\mathbf{x} - P^j| \leq C\varepsilon^{1/2}\}$ zone (or equivalently, for $|\xi^j| = O(\varepsilon^{-1/2})$), we derive the above-mentioned conditions

$$\mathbf{w}^{0j}(\xi) = d(P^j)a^0 + R_j\mathbf{T}(\xi) + o(|\xi|^{-1}), |\xi| \to \infty. \tag{2.12}$$

3. SOLUTION OF THE MODEL SIGNORINI EQUATION

If the inequality $a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j) \geq 0$ holds, it is easy to see that problem (2.9), (2.10), (2.12) is satisfied by the constant vector $\mathbf{w}^{0j} = d(P^j)a^0$ and R_j vanishes.

We now consider the case when $a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j) < 0$. We denote by $\mathbf{c}_j$ the capacity $\operatorname{cap}(\omega_j)$, of the set $\omega_j = \{\xi : \xi' \in \omega_j,\ \xi_3 = 0\}$, and by Y_j the corresponding capacitive potential (see [5])

$$-\Delta_\xi Y_j(\xi) = 0, \xi \in \mathbf{R}^3 \setminus \overline{\omega}_j; \quad Y_j(\xi) = 1, \xi \in \omega_j \tag{3.1}$$
$$Y_j(\xi) = \mathbf{c}_j|\xi|^{-1} + O(|\xi|^{-2}), |\xi| \to \infty$$

In accordance with the Papkovich–Neuber representation, the capacitive potential gives the solution to the contact problem of the indentation (without turning) of a smooth punch with a flat ω_j-shaped base into an elastic half-space $\xi_3 \geq 0$ to unit depth (see [6, 7])

$$W_i^j(\xi) = \alpha[(\alpha^{-1} - 1)\int_{\xi_3}^{\infty} \partial_i Y_j(\xi',\zeta)d\zeta\ \xi_3\partial_i Y_j(\xi)], \quad i = 1,2$$
$$W_3^j(\xi) = Y_j(\xi) - \alpha\xi_3\partial_3 Y_j(\xi), \partial_i = \partial / \partial\xi_i$$

The pressure at the boundary of the half-space produced by the punch is calculated from the value of the normal derivative of the function Y_j and is equal to

$$-\sigma_{33}^j(\xi',0) = -2\mu\alpha\partial_3 Y_j(\xi',+0)$$

The pressure at the base of the punch is positive (by the maximum principle $\partial_3 Y_j(\xi',\ +0) < 0$ when $\xi' \in \omega_j$), and has a root singularity at the edge of the contact surface

$$Y_j(\xi) = 1 + K_j(\tau)\rho^{1/2}\cos\tfrac{1}{2}\varphi + \tilde{Y}(\xi), \left|\nabla_\xi^k \tilde{Y}_j(\xi)\right| \leq c_\delta\rho^{\delta-k+3/2} \tag{3.2}$$

Here τ is the arc length along $\partial\omega_j$ and (ρ, φ) are polar coordinates in the planes perpendicular to $\partial\omega_j$. K_j is a positive function in $C^\infty(\partial\omega_j)$, and $0 < \delta$ is otherwise arbitrary.

If the punch, which is loaded with a point force parallel to the ξ_3 axis, is to undergo only a translational displacement, it is necessary and sufficient for the line of action of the force to coincide with the line

$$\xi_i = \xi_i^{\circ j} \equiv \left(\int_{\omega_j} \partial_3 Y_j(\eta, +0) d\eta \right)^{-1} \left(\int_{\omega_j} \eta_i \partial_3 Y_j(\eta, +0) d\eta \right), \quad i = 1, 2$$

The point $(\xi_1^{\circ j}, \xi_2^{\circ j})$ is called the centre of pressure of the plane shape ω_j. If the origin of coordinates is displaced to the point $(\xi_1^{\circ j}, \xi_2^{\circ j}, 0)$ then the asymptotic term Y_j acquires at infinity the form $Y_j(\xi) = \mathbf{c}_j |\xi|^{-1} + O(|\xi|^{-3})$, (cf. (3.1)). Below we shall assume that the points P_j coincide with the centres of pressure ω_j.

Conditions (2.9)–(2.12) are therefore satisfied if

$$\mathbf{w}^{0j}(\boldsymbol{\xi}) = d(P^j)a^0 - (a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j))\mathbf{W}^j(\boldsymbol{\xi}) \tag{3.3}$$

Comparing the asymptotic behaviour of the vector (3.3) at infinity with formula (2.12) we obtain

$$R_j = -\kappa_j (a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j)), \quad \kappa_j \equiv 4\mu\alpha c_j$$

Let $(t)_+ = (t + |t|)/2$ be the positive part of the number $t \in \mathbf{R}$. The terms in the expansions (2.1) and (2.2) are found apart from some constants. The vector a^1 is calculated at the next step of the construction of the asymptotic form, and to determine the reaction forces $R_1, \ldots, R_J$ and settlement parameters a_3^0, a_4^0, a_5^0, in addition to the equilibrium conditions (2.8), the conditions

$$a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j) \geqslant 0 \Rightarrow R_j = 0$$

$$a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j) < 0 \Rightarrow R_j = -\kappa_j (a_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j))$$

also appeared.

In short notation these conditions are

$$R_j = \kappa_j(-a_3^0 + 2^{-1/2}(a_5^0 x_1^j - a_4^0 x_2^j))_+, \; j = 1, \ldots, J \tag{3.4}$$

We call relations (3.4) together with (2.8) the limiting algebraic problem. We emphasize that after solving this problem the terms $\mathbf{w}^{0j}$ of the inner expansions (2.2) are recovered, together with the term $\mathbf{v}^0(\mathbf{x}) = d(\mathbf{x})a^0$ from (2.1). Finally, the second term in (2.7) of the outer expansion is determined, apart from the term $d(\mathbf{x})a^1$. Restricting ourselves to constructing the dominant terms of the asymptotic form, we now put $a^1 = 0$.

4. THE SOLVABILITY OF THE LIMITING ALGEBRAIC PROBLEM

We will formulate and prove sufficient conditions for the existence and uniqueness of the solution of problem (2.8), (3.4).

Assertion 1. Suppose the following two conditions are satisfied: (1) three of the points P^j, $j = 1, \ldots, J$ $(J \geqslant 3)$ do not lie on the same straight line, and (2) x^0 is an internal point of $\mathbf{P}$. Then the limiting algebraic problem has a unique solution.

Proof. We define the continuous non-linear operator $N: \mathbf{R}^3 \to \mathbf{R}^3$ by the formula $N(a) = XM[X'a]_+$, where X is a $3 \times J$ matrix with columns $(1, x_1^j, x_2^j)'$, M is a diagonal $J \times J$ matrix $\operatorname{diag}\{\kappa_1, \ldots, \kappa_J\}$ and $[q]_+ = ((q_1)_+, \ldots, (q_{J+})')$ for a vector q in $\mathbf{R}^J$. Problem (2.8), (3.4) is equivalent to solving the operator equation

$$N(\mathbf{a}) = h \equiv (-F_3(1), -x_1^\circ F_3(1), -x_2^\circ F_3(1))' \tag{4.1}$$

We denote by $\langle , \rangle$ the scalar product in $\mathbf{R}^3$. The limiting algebraic problem can also be reduced to the problem of minimizing the functional

$$\Phi(\mathbf{a};h) = \tfrac{1}{2}\langle N(\mathbf{a}),\mathbf{a}\rangle - \langle h,\mathbf{a}\rangle$$

in $\mathbf{R}^3$.

For any non-zero $\mathbf{a} \in \mathbf{R}^3$ the point $e_a = \|a\|^{-1}\mathbf{a}$ lies on the unit sphere S and we have the representation

$$\Phi(\mathbf{a};h) = \tfrac{1}{2}\|\mathbf{a}\|^2 \langle XM[X^t e_a]_+, e_a\rangle - \|\mathbf{a}\|\langle h, e_a\rangle$$

Because the function $\mathbf{R}^1 \ni t \mapsto (t)_+ t$ is convex and M is a diagonal matrix, the function Φ is convex on $\mathbf{R}^3$, i.e.

$$\Phi(\lambda\mathbf{a}^1 + (1-\lambda)\mathbf{a}^2 \leqslant \lambda\Phi(\mathbf{a}^1) + (1-\lambda)\Phi(\mathbf{a}^2) \forall \mathbf{a}^1, \mathbf{a}^2 \in \mathbf{R}^3, \forall\lambda \in (0,1)$$

The relation $\langle XM[X^t e]_+, e\rangle \geqslant 0$ is also satisfied.

We will denote by K_N the kernel of the restriction of the operator N to S

$$K_N = \{e \in S: N(e) = 0\} = \{e \in S: \langle XM[X^t e]_+, e\rangle = 0\}$$

The inclusion $e \in K_N$ is equivalent to the system of inequalities

$$e_1 + e_2 x_1^j + e_3 x_2^j \leqslant 0, \quad j = 1, \ldots, J \tag{4.2}$$

Consequently, K_N is a closed subset of S.

Condition (4.2) ensures that all the points P^j lie in the single closed half-plane $e_1 + e_2 x_1 + e_3 x_2 \leqslant 0$, which also contains the polygon $\mathbf{P}$. According to the first assumption the convex envelope of the points P^j does not degenerate into a line interval, and the interior of $\mathbf{P}$ therefore lies entirely in the open half-plane $e_1 + e_2 x_1 + e_3 x_2 \leqslant 0$. Hence, if $F_3(1) < 0$ (see condition (1.6)), then $-\langle h, e\rangle > 0$ for any $e \in K_N$. Because K_N is closed, there is an open set $Q \subset S$ such that $Q \supset K_N$ and for $\forall e \in \overline{Q}$ the inequality $-\langle h, e\rangle \geqslant \beta_1 > 0$ holds. We put

$$\max_{e \in S}\langle h, e\rangle = \beta_2 > 0, \quad \min_{e \in S \setminus Q}\langle N(e), e\rangle = \beta_3 > 0$$

The following limits hold

$$\Phi(\mathbf{a};h) \geqslant \|\mathbf{a}\|\beta_1, \quad \forall \|\mathbf{a}\|^{-1}\mathbf{a} \in \overline{Q}$$
$$\Phi(\mathbf{a};h) \geqslant \tfrac{1}{2}\|\mathbf{a}\|^2\beta_3 - \|\mathbf{a}\|\beta_2, \forall \|\mathbf{a}\|^{-1}\mathbf{a} \in S \setminus \overline{Q}$$

For each $T > 0$ a $t > 0$ exists (depending only on T) such that for all $\mathbf{a}$ in $\mathbf{R}^3$ satisfying $\|a\| \geqslant t$ we have the inequality $\Phi(\mathbf{a}, h) \geqslant T$. Indeed, it is sufficient to put

$$t = \max\{\beta_1^{-1}T, \beta_3^{-1}(\beta_2 + (\beta_2^2 + 2\beta_3 T)^{1/2})\}$$

Hence, $\lim\Phi(\mathbf{a}) = \infty$ as $\|\mathbf{a}\| \to \infty$, and according to a well-known theorem of convex analysis (see e.g. Section 2.2 of [2]), solutions of problem (4.1) exist and generate a convex subset of $\mathbf{R}^3$.

The uniqueness of the solution is also proved by applying the above theorem, because the functional is strictly convex on the set of solutions by virtue of the diagonality of M and the inequality

$$(\lambda t + (1-\lambda)s)_+(\lambda t + (1-\lambda)s) < \lambda t_+ t + (1-\lambda)s_+ s$$
$$\forall s, t \in \mathbf{R}, s \neq t, s > 0; \quad \lambda \in (0,1)$$

5. THE PROPERTIES OF SOLUTIONS OF THE LIMITING ALGEBRAIC PROBLEM

Let the column $\mathbf{a}^0 \in \mathbf{R}^3$ be a solution of Eq. (4.1). Renumbering the points P^j if necessary, we can assume that the positive components of the vector $M[X^t\mathbf{a}^\circ]_+$ in $\mathbf{R}^J$ determine the set of reaction forces $R_1, \ldots, R_{J^\circ}$ $(J^\circ \leq J)$. The following obvious assertion contains a necessary condition for the existence of a solution of Eq. (4.1)

Assertion 2. If a solution of problem (2.8), (3.4) exists, then $x^\circ = (x_1^\circ, x_2^\circ)$ lies in the convex envelope $\mathbf{P}^\circ$ of the points $\mathbf{P}^1, \ldots, \mathbf{P}^{J^\circ}$.

Corollary. If x° does not belong to $\mathbf{P}$, then the limiting algebraic problem has no solution.

The number of supports and their relative position naturally influence the properties of the limiting problem, and a useful characteristic turns out to be the rank of the matrix X. A single support corresponds to $\operatorname{rank} X = 1$. For several points lying along a single straight line $\operatorname{rank} X = 2$. If however $\operatorname{rank} X = 3$, then, firstly, $J \geq 3$, and secondly, there are at least three points not lying along the same straight line.

In the cases $\operatorname{rank} X = 1, 2$, $x^\circ \in \mathbf{P}$ or $\operatorname{rank} X = 3$, $x^\circ \in \partial\mathbf{P}$ Eq. (4.1) has infinitely many solutions. We note that when $x^\circ \in \mathbf{P}$ the point x° is an internal point of $\Gamma(\varepsilon)$ for all $\varepsilon > 0$ and the solution of the original problem (1.3)–(1.5) is unique. This paradox is explained by the fact that in this situation the asymptotic construction used previously needs to be corrected. Here it is appropriate to describe the equilibrium position of the body Ω on supports $\Phi(\varepsilon)$ as being unstable in the asymptotic sense.

We will indicate some properties of solutions of the limiting algebraic inequality in the non-trivial case.

Assertion 3. Suppose that $\operatorname{rank} X = 3$ and x° is an inner point of $\mathbf{P}$. If $\mathbf{a}^0$ is a solution of problem (4.1), then, firstly, $J^\circ \geq 3$, secondly the points $\mathbf{P}^1, \ldots, \mathbf{P}^{J^\circ}$ lie in the open half-plane $\mathbf{a}_1^0 + \mathbf{a}_2^0 x_1 + \mathbf{a}_3^0 x_2 > 0$, and thirdly, x° lies in the interior of the supporting polygon $\mathbf{P}^\circ$.

To prove these assertions (see also Section 112 of [8]) it is convenient to assign positive weights κ_j to the points P^j and introduce a system of coordinates attached to the principal axes of inertia of the system of material points $P^1, \ldots, P^J$ or $P^1, \ldots, P^{J^\circ}$. This mechanical analogy suggests an answer to the question of the conditions under which the body will be certain to rest on all the supports.

Assertion 4. If, under the conditions of Assertion 3, the point x° is contained in a sufficiently small neighbourhood of the centre of mass of the system of material points $P^1, \ldots, P^J$ then $J^\circ = J$.

We note that we have in passing established the necessity of the conditions formulated in Assertion 1.

6. JUSTIFICATION OF THE ASYMPTOTIC FORM

We will additionally assume that the solution of the limiting algebraic problem possesses the following property

$$\bar{a}_3^0 + 2^{-1/2}(a_4^0 x_2^j - a_5^0 x_1^j) \neq 0, \quad j = 1, \ldots, J \tag{6.1}$$

We have thus eliminated the case of the body touching a support without a positive reaction force. Let J° be the number from the preceding section. One can verify that in the original Signorini problem (1.4)–(1.6) we have the boundary conditions

$$\begin{aligned} &u_3(\varepsilon, \mathbf{x}) = 0, \quad \mathbf{x} \in \omega_j(\varepsilon), j = 1, \ldots, J^\circ \\ &\sigma_{33}(\mathbf{u}; \varepsilon, \mathbf{x}) = 0, \mathbf{x} \in \omega_j(\varepsilon), j = 1 + J^\circ, \ldots, J \end{aligned} \tag{6.2}$$

We will sketch a proof of this fact. We take the vector

$$\mathbf{U}(\varepsilon,\mathbf{x}) = \chi_0(\varepsilon,\mathbf{x})[\varepsilon^{-1}d(\mathbf{x})a^0 + \mathbf{v}^1(\mathbf{x})] + \varepsilon^{-1}\sum_{j=1}^{J^\circ} \chi_j(\mathbf{x})\mathbf{w}^{0j}(\boldsymbol{\xi}^j) -$$
$$\varepsilon^{-1}\chi_0(\varepsilon,\mathbf{x})\sum_{j=1}^{J^\circ} \chi_j(\mathbf{x})[d(P^j)a^0 + R_j\mathbf{T}(\boldsymbol{\xi}^j)] \tag{6.3}$$

to be an asymptotic approximation to the solution $\mathbf{u}(\varepsilon, \mathbf{x})$. Here a^0, $\mathbf{v}^1$, $\mathbf{w}^{0j}$, R_j are the quantities defined previously, and χ_j, χ_j' are cut-off functions on $C_0^\infty(\mathbf{R}^3)$, with $\chi_j' = 1$ near the set ω_j, $\chi_j = 1$ in a neighbourhood of the point P^j, $\chi_j = 0$ near $\partial\Omega\setminus\Sigma$ and $\chi_j = 0$ around $P^q (q \neq j)$

$$\chi_0(\varepsilon,\mathbf{x}) = 1 - \chi'_1(\varepsilon^{-1}(\mathbf{x}-P^1)) - \ldots - \chi'_{J^\circ}(\varepsilon^{-1}(\mathbf{x}-P^{J^\circ})). \tag{6.4}$$

We will clarify the construction of (6.3). The first term on the right is the leading term of the outer expansion; with the help of the cut-off (6.4) it is cancelled outside the zone of action of this expansion (in the immediate vicinity of P^j, $j=1, \ldots, J^\circ$). The second term contains components of the inner expansions; due to the cut-offs χ_j the boundary layers are localized near the P^j. Finally, the terms of the asymptotic representations (2.11) and (2.12) are the same (having undergone matching) and are used twice in (6.3): both in the first and second expressions on the right. This doubling is removed by the subtraction of the third term.

We substitute (6.3) into the (linear!) problem (1.4), (1.5), (6.2); we denote the exact solution of the latter by $\mathbf{u}^l$. By the constructions performed the errors in equalitites (1.4) are small, and the boundary conditions (1.5) and (6.2) are completely satisfied.

With the help of results from [9, 10] we obtain an estimate of the difference $\mathbf{r} = \mathbf{u}^l - \mathbf{U}$ in some weighted space. In particular, from such an estimate it follows that the traces of the r_3 on $\omega_j(\varepsilon)$, $j = 1 + J^\circ, \ldots, J$ are small, from which, by (6.1), the inequalities $u_\varepsilon^l = U_3 - r_3 > 0$ on $\omega_j(\varepsilon)$, $j = 1 + J^\circ, \ldots, J$, follow for sufficiently small ε. This estimate cannot guarantee the smallness of the traces of $\sigma_{33}(\mathbf{r})$ because of the singularities of the stresses at the edges $\partial\omega_j(\varepsilon)$. However due for formulae (6.1) and (3.2) (recalling that $K_j > 0$) the relations $\sigma_{33}(\mathbf{U}) + \sigma_{33}(\mathbf{r}) < 0$ hold on $\omega_j(\varepsilon)$, $j = 1 + J^\circ$. The inequalities from (1.6) are therefore satisfied by $\mathbf{u}^l$, which means that $\mathbf{u}^l$ is a solution of the Signorini problem (1.4)–(1.6). Finally, this estimate justifies the asymptotic behaviour $\mathbf{u} \sim \mathbf{U}$, and also verifies the outer expansion (2.1) and the inner expansion (2.2) of the solution $\mathbf{u}(\varepsilon, \mathbf{x})$.

Note that the arbitrariness (an element from the lattice $\mathbf{R}''$) in the choice of solution in all the problems is the same and hence has been ignored.

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